MIMO RADAR PERFORMANCE ANALYSIS UNDER K-DISTRIBUTED CLUTTER

Xin Zhang*, Mohammed Nabil El Korso** and Marius Pesavento*

*Communication Systems Group, Technische Universität Darmstadt, Darmstadt, Germany
**Laboratoire Energétique Mécanique Electromagnétique, Université Paris Ouest Nanterre La Défense, Ville d’Avray, France

ABSTRACT

The resolvability of two closely-spaced signals is an important performance measure for parametric estimation problems. In this paper we investigate the so-called resolution limit (RL) in a MIMO radar context, i.e., the minimum angular separation required to resolve two closely-spaced targets. Due to the limited number of elementary scatterers, the Gaussian modeling of the clutter is inappropriate. In our analysis, we consider a K-distributed clutter which is a well-established approximation of the real clutter. Finally, our RL’s expression reveals a number of insightful properties that are discussed in detail and, numerical examples are provided to corroborate the theoretical analysis.

Index Terms— K-distributed clutter, performance analysis, Cramér-Rao bound, resolution limit, MIMO radar.

1. INTRODUCTION

Multiple-input multiple-output (MIMO) radar, by employing multiple, spatially distributed transmitters and receivers to achieve waveform diversity [1–3], has the advantage to fundamentally improve the performance of communication systems. In the past decade, immense academic interest has been created concerning the investigation of new algorithms for target localization in MIMO systems and its performance analysis, in terms of lower bounds for parameter estimation accuracy and resolvability [1, 2, 4–7]. The resolvability problem of two closely-spaced signals is one of the most significant performance measures in estimation theory. In regard to this subject, a number of recent works, e.g., [6–8] have applied the concept of the resolution limit (RL) [9–11] in the MIMO context. In all of these works, however, the radar clutter is exclusively modeled as Gaussian processes, an assumption which can be, unfortunately, unrealistic in real life application. This is due to the fact that, the Gaussian modeling of the clutter, notwithstanding its popularity, loses immediately its validity, when the assumption required for the central limit theorem, i.e., that the received clutter results from a large number of independent and identically distributed (i.i.d.) elementary scatterers, does not hold. This is for example the case in the context of low-grazing-angle and/or high-resolution radar [12–14]. For MIMO radar, which generally has a considerably higher resolvability than its traditional phased-array counterpart, Gaussian modeling of the clutter is often considered inadequate due to the limited number of i.i.d. elementary scatterers present in each cell.

To grapple with these non-Gaussian clutter cases, numerous clutter models have been devised, out of which the K-distributed clutter [12, 15, 16] has become a popular one, due to its ability to characterize nonhomogeneous ground clutter as well as spiky sea clutters, as supported by experimental data measurements [12, 16, 17]. The K-distributed clutter is the product of two components: the square root of a gamma process (called the texture) accounting for the local scattering, and a complex Gaussian process (called the speckle) describing the local power changing. A K-distributed clutter is thus fully characterizable by its texture parameters (i.e., the shape and scale parameter of the gamma distribution) and its speckle covariance matrix. Abounding works already exist to investigate detection/estimation algorithms under the K-distributed clutter [12, 15, 16], however, to the resolvability problem of radar targets under the K-distributed clutter, despite all its theoretical significance, no available work in the current literature has yet been dedicated. It is our aim to fill this gap, by addressing the following task: “What is, in a MIMO radar context under the K-distributed clutter, the minimum angular separation required, under which two targets can still be correctly resolved?”

The aforementioned concept of the RL serves as the keystone in approaching such a task. The RL is commonly defined as the minimum distance w.r.t. the parameter of interest that allows distinguishing between two closely-spaced sources [9–11] and can be described, respectively, on the basis of three distinctive theories: the mean null spectrum analysis theory [18], the detection theory [10, 19, 20], and finally, the estimation theory, which exploits the definition of the Cramér-Rao bound (CRB) [9, 21, 22]. One prevalent criterion based on the estimation theory, proposed by Smith [9], states that two targets are resolvable if the distance between the targets (w.r.t. the parameter of interest) is greater than the standard deviation of the distance estimation. Smith’s criterion considers the coupling between the parameters and thus is preferable to other estimation theory based criteria, e.g., the ones of [21, 23]. Furthermore, the RL obtained in Smith’s sense has been proved to be closely related to the detection theory based RL [11]. Finally, Smith’s criterion enjoys generality as, unlike the mean null spectrum analysis approach, it is not designed for certain specific high-resolution algorithms. In view of these merits, in this paper we focus on Smith’s criterion, to evaluate the RL between two closely-spaced targets in a MIMO radar system under the K-distributed clutter, to inspect its various properties, and to reveal emphatically how the choices of the texture parameters can exert a decisive influence on the behavior of the RL.

2. OBSERVATION MODEL

Consider a colocated MIMO radar system with uniform linear arrays (ULAs) both at the transmitter and the receiver, illuminating two far-field, narrowband, point targets [1]. The radar output, without matched filtering, for the fth radar pulse in a coherent processing interval (CPI) is given as the following vector form [7]:
Furthermore, for the convenience of later derivation we define the direction-of-arrival(DOA)/direction-of-departure(DOD) of the received clutter vectors for the th pulse, respectively; and the transmitting and receiving steering vectors are defined as \( \alpha \) l, respectively; the transmitting and receiving steering vectors are defined as \( \alpha \) l. Additionally, the number of snapshots per pulse and the number of pulses per CPI, and the remaining part of the paper.

3.1. Score Functions

In order to evaluate the RL in Smith’s sense, we have to first obtain the CRB w.r.t. the parameter \( \Delta \). The latter, denoted as CRB(\( \Delta \)), is related to the Fisher Information Matrix (FIM) corresponding to \( \xi \), denoted by \( F \), as CRB(\( \Delta \)) = CRB \((\xi_1)\) = \([F^{-1}]_{1,1}\).

3.1. Score Functions

The computation of the entries of \( F \) involves partial derivatives of the log-likelihood function \( \Lambda \) w.r.t. all elements of \( \xi \) (also known as the score functions) and is given as:

\[
[F]_{i,j} = E \left\{ \frac{\partial \Lambda}{\partial [\xi_i]} \frac{\partial \Lambda}{\partial [\xi_j]} \right\}.
\]
Introduce the parameter sets \( \beta \in \{ \Delta, \pi_1, \overline{\pi}_1, \pi_2, \overline{\pi}_2 \} \) and \( \varsigma \in \{ \phi, \varsigma \} \). The partial derivatives \( \partial \Lambda / \partial [\xi] \), are straightforwardly computed as:

\[
\frac{\partial \Lambda}{\partial \beta} = -\sum_{i=1}^{T} \mu_{\gamma(i)}^2 \cdot \left( \gamma^H(t) \Sigma^{-\frac{1}{2}} \partial V(t) \partial \beta \Sigma^{-\frac{1}{2}} \gamma(t) \right),
\]

\( \beta \in \{ \Delta, \pi_1, \overline{\pi}_1, \pi_2, \overline{\pi}_2 \} \); \hspace{1cm} (12a)

\[
\frac{\partial \Lambda}{\partial [\xi]} = -\sum_{i=1}^{T} \mu_{\gamma(i)}^2 \cdot \left( \gamma^H(t) \Sigma^{-\frac{1}{2}} \partial \xi \Sigma^{-\frac{1}{2}} \gamma(t) \right) - T \text{tr} \left( \Sigma^{-1} \partial \xi \Sigma^{-1} \right), \quad i = 1, \ldots, N^2, \]

\[
\frac{\partial \Lambda}{\partial \varsigma} = \sum_{t=1}^{T} \mu_{\gamma(t)}, \quad \varsigma \in \{ \phi, \varsigma \}; \hspace{1cm} (12c)
\]

where \( \text{tr} \{ \cdot \} \) denotes the trace of a matrix, and:

\[
\mu_{\gamma(i)}^2 = \left( \frac{\partial Q_N(\gamma(t))^2}{\partial \gamma(t)} \right) f_N(\|\gamma(t)\|, \phi, \varsigma) \]

\[
= \int_{0}^{\infty} \tau^{-N-1} \tau^{-N}(t) e^{-\frac{\gamma(t)^T}{2\tau}} p_{\tau}(\tau(t); \phi, \varsigma) d\tau(t); \hspace{1cm} (13a)
\]

\[
\mu_{\varphi} = \left( \frac{\partial Q_N(\gamma(t))^2}{\partial \varphi} \right) f_N(\|\gamma(t)\|, \phi, \varsigma) \]

\[
= \int_{0}^{\infty} \tau^{-N}(t) e^{-\frac{\gamma(t)^T}{2\tau}} p_{\tau}(\tau(t); \phi, \varsigma) d\tau(t), \quad \varsigma \in \{ \phi, \varsigma \}; \hspace{1cm} (13b)
\]

### 3.2. Entries of \( F \)

The entries of \( F \) are obtained by substituting Eq. (12a)-(12b) into Eq. (11). We use the same methodology as in [25, 26] to obtain the analytical expressions of the FIM’s entries, and find that \( F \) has the following block-diagonal structure:

\[
F = \begin{bmatrix}
\Phi & 0_{5 \times (N^2+2)} & \Xi \\
o_{(N^2+2) \times 5} & 0_{(N^2+2) \times (N^2+2)} & \Xi
\end{bmatrix},
\]

where

\[
\Phi = \begin{bmatrix}
f_{\beta \Delta} & f_{\beta \pi_1} & f_{\beta \pi_1} & f_{\beta \pi_2} & f_{\beta \pi_2} \\
f_{\beta \pi_1} & f_{\beta \pi_1} & f_{\beta \pi_2} & f_{\beta \pi_2} & f_{\beta \pi_2} \\
f_{\beta \pi_1} & f_{\beta \pi_2} & f_{\beta \pi_2} & f_{\beta \pi_2} & f_{\beta \pi_2} \\
f_{\beta \pi_1} & f_{\beta \pi_2} & f_{\beta \pi_2} & f_{\beta \pi_2} & f_{\beta \pi_2} \\
f_{\beta \pi_1} & f_{\beta \pi_2} & f_{\beta \pi_2} & f_{\beta \pi_2} & f_{\beta \pi_2}
\end{bmatrix},
\]

is a 5 \times 5 symmetric submatrix of \( F \) containing FIM entries w.r.t. the target parameters (\( \Delta, \pi_1 \), and \( \overline{\pi}_1, i = 1, 2 \)). The expressions of the entries of \( \Phi \) are obtained by invoking Lemma 2 and Lemma 3 proved in [26], and by considering the circularity property that \( E \{ x(i) x^T(j) \} = 0 \) for all \( i, j = 1, \ldots, T \) in Eq. (6) and that the speckle is uncorrelated with the texture (hence \( E \{ \gamma(t) \gamma^T(t) \} = 0 \)), leading to

\[
f_{\beta_1 \beta_2} = \frac{2}{N} E \left( \sum_{t=1}^{T} \mu_{\gamma(i)}^2 \cdot |\gamma(t)|^2 \cdot \text{tr} \left( \frac{\partial V(t) \partial V^H(t)}{\partial \beta_1 \partial \beta_2} \Sigma^{-1} \right) \right), \hspace{1cm} (16)
\]

\[
\beta_1, \beta_2 \in \{ \Delta, \pi_1, \overline{\pi}_1, \pi_2, \overline{\pi}_2 \}, \quad i = 1, 2;
\]

\[
\kappa = E \{ |\gamma(t)|^2 \cdot \mu^2_{\gamma(t)} \}; \quad (17)
\]

where \( \kappa \) is the CRLB (18a), (18b) and (18c) that the entries of \( \Phi \) are highly non-linear w.r.t. \( \Delta \), meaning that all attempts of finding an analytical solution of Eq. (19) are generally difficult without resort to certain means of approximation (e.g., the Taylor expansion), i.e., without impairing the accuracy of the value of \( \delta \). This fact, combined with the mathematical difficulty of inverting the 5 \times 5 matrix \( \Phi \), makes such an attempt, if still feasible, undesirable. Therefore we shall numerically evaluate the value of \( \delta \), to avoid the aforementioned drawbacks and to retain the true, unapproximated value of \( \delta \). The results of our evaluation, with all their significance and implications, will be presented in Section 4.

### 3.3. Smith Equation

Now, in consonance with Smith’s criterion [9], these two targets can be angularly resolved if \( \Delta \) is greater than the standard deviation of the estimate of \( \Delta \); while the later, under mild conditions [27], can be approximated by \( \sqrt{\text{CRLB}(\Delta)} \). Thus, the RL of the two targets in our model, henceforth denoted by \( \delta \), is equal to the (approximated) standard deviation of its estimate, i.e., \( \delta \) fulfills the following expression (also known as the Smith equation):

\[
\delta^2 = \text{CRLB}(\delta). \hspace{1cm} (19)
\]

It is noticeable from Eq. (18a), (18d) and (18e) that the entries of \( \Phi \) are highly non-linear w.r.t. \( \Delta \), meaning that all attempts of finding an analytical solution of Eq. (19) are generally difficult without resort to certain means of approximation (e.g., the Taylor expansion), i.e., without impairing the accuracy of the value of \( \delta \). This fact, combined with the mathematical difficulty of inverting the 5 \times 5 matrix \( \Phi \), makes such an attempt, if still feasible, undesirable. Therefore we shall numerically evaluate the value of \( \delta \), to avoid the aforementioned drawbacks and to retain the true, unapproximated value of \( \delta \). The results of our evaluation, with all their significance and implications, will be presented in Section 4.

### 3.4. Calculation of \( \kappa \)

A crucial but tricky point in evaluating the value of \( \delta \) is the calculation of \( \kappa \) in Eq. (17), as the latter is devoid of any closed-form...
expression, thus can only be gauged by numerical means. By applying (C.48) and (C.49) in [26]:

\[
\kappa = \frac{\int_{a}^{b} \mu_{2,T}^2(\tau) \cdot q_{\alpha} \left( \frac{[\gamma(\tau)]^2}{\phi, \psi} \cdot |\gamma(\tau)|^{2N+1} \right) d|\gamma(\tau)|}{q_{\alpha} \left( |\gamma(\tau)|^2, \phi, \psi \right) \cdot |\gamma(\tau)|^{2N+1} d|\gamma(\tau)|}
\]

(20)

Substituting Eq. (9) and (13a) into Eq. (20) and then using the generalized Gauss-Laguerre quadrature [28] for all layers of integrations, we obtain:

\[
\kappa = \sum_{m_{m1} = 1}^{\infty} \sum_{m_{m4} = 1}^{\infty} \left( \frac{x_{m1}^{[0,1]} \exp \left( -\frac{x_{m1}}{2} \right) x_{m2}^{[N,2]} \exp \left( -\frac{x_{m2}}{2} \right) x_{m3}^{[N,3]} \exp \left( -\frac{x_{m3}}{2} \right) x_{m4}^{[N,4]} \exp \left( -\frac{x_{m4}}{2} \right) \right) \phi^{x_{m1}} \psi^{x_{m4}}
\]

(21)

where \(x_{m1}, i = 1, \ldots, 5\) and \(w_{m1}, i = 1, \ldots, 5\) represent the abscissae and the weights of the generalized Gauss-Laguerre quadrature, respectively; \(M_{1N}^{[2N+1]}\), \(M_{2N}^{[0,1]}\), etc. denote the quadrature orders; with the subscript of each representing the respective parameter of the corresponding abscissa and weight (e.g., \(M_{1N}^{[2N+1]}\) means that \(x_{m1}\) and \(w_{m1}\) have \(2N + 1\) as their parameter). In our simulations, the values of these quadrature orders are empirically chosen as \(M_{1N}^{[2N+1]} = 212, M_{2N}^{[0,1]} = 214, M_{3N}^{[0,1]} = 196, M_{4N}^{[2N+1]} = 200, \) and \(M_{5N}^{[0,1]} = 168, \) to generate the most accurate results.

4. NUMERICAL SIMULATIONS

Consider a MIMO radar with \(M = 6\) sensors at the transmitter and \(N = 8\) at the receiver, both with half-wave length inter-element spacing; the snapshot number per CPI \(T = 500\); the complex coefficients \(\alpha_1 = 1 + j\) and \(\alpha_2 = 1 - j\); all the real and imaginary parts of the target source vectors \(s(t)\)'s entries are randomly generated within \([-1, 1]\); while the entries of the speckle covariance matrix \(\Sigma\) are generated by: \(\Sigma_{m,n} = \sigma^2 \cdot 0.9^{m-n} \cdot 2^{(m-n)}\), \(m, n = 1, \ldots, N\), where \(\sigma^2\) denotes the speckle power factor [29].

Note that the texture and speckle are mutually uncorrelated, and that for the K-distributed clutter, the mean of the texture \(E\{\tau(t)\} = \phi, \psi\), the target-to-clutter ratio (TCR) can be formulated as:

\[
TCR = \frac{\sum_{t=1}^{T} |s(t)|^2}{T \cdot E\{n^H(n)\}} = \frac{\sum_{t=1}^{T} |s(t)|^2}{T \cdot \phi \cdot \psi \cdot \text{tr}(\Sigma)}
\]

(22)

- We first investigate the respective impact of the texture parameters \(\phi\) and \(\psi\) on \(\delta\), by fixing only one texture parameter and changing the other. In Fig. 1, we plot \(\delta\) vs. TCR with fixed \(\psi\) and various \(\phi\) and find that \(\delta\) increases notably with \(\phi\). This is because the rise of the shape parameter \(\phi\) makes the clutter more heavy-tailed, thus a larger portion of the clutter power decentralized, leading to a deteriorated \(\delta\).

- In Fig. 2 we do the converse, by plotting \(\delta\) vs. TCR with different \(\psi\) and fixed \(\phi\). Fig. 2 shows, quite interestingly, that \(\delta\) remains substantially the same for, thus could be regarded as independent of, the different choices of \(\psi\).

- In Fig. 1 & Fig. 2, where we vary only one texture parameter and fix the other, we have thereby also changed the texture power (and thus also the speckle power, for each specific TCR). In Fig. 3, we fix the texture power \(E\{\tau(t)\} = \phi = 4\) and choose several pairs of \(\phi\) and \(\psi\) corresponding to this power and examine \(\delta\) in this context. We also plot \(\delta\) under Gaussian clutter for comparison (for which \(\tau(t) = \hat{\delta}(\tau(t) - 1)\) with \(\hat{\delta}(\cdot)\) denoting the Dirac delta function, leading to \(\kappa = \infty\)). Fig. 3 shows that \(\delta\) rises with a pair of increasing \(\phi\) and decreasing \(\psi\), at the same time, \(\delta\) cannot rise infinitely, and is upper-bounded by \(\delta\) under Gaussian clutter.

5. SUMMARY

We have in this paper investigated the resolvability problem in a MIMO context under the K-distributed clutter, by applying the concept of the RL in Smith’s sense. We have firstly derived the FIM expressions, and then by virtue of the Smith equation we have numerically obtained the value of the RL \(\delta\). Our simulation concentrates on the impact of texture parameters on \(\delta\), and reveals that \(\delta\) increases with \(\phi\) but remains unreactive to the change of \(\psi\). Finally, our results show that \(\delta\) under all K-distributed clutters, regardless of their texture parameters, is smaller than \(\delta\) under the Gaussian clutter.
References


