Rooting-Based Harmonic Retrieval Using Multiple Shift-Invariances: The Complete and the Incomplete Sample Cases

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Abstract—In the present paper, we propose a novel method for estimating one-dimensional damped and undamped harmonics. Our method utilizes the multiple shift-invariance property comprised in the signal model. We develop a new rank-reduction estimator which is formed as the weighted sum of the individual matrix polynomials obtained from individual shift-invariance equations. The uniqueness conditions for the proposed rank-reduction criteria are derived under the assumption that all samples are available. Moreover, a novel technique for the incomplete data case, where some samples are missing, is presented. In this case, the rank-reduction estimator may suffer from ambiguities. To overcome this problem, we propose an extension of the rank-reduction estimator that is based on polynomial intersection and the properties of the Sylvester matrix. The latter algorithm yields unique estimates of the damped harmonics. The proposed high-resolution techniques are search-free and therefore, they enjoy moderate computational complexity.

Index Terms—Direction of arrival estimation, harmonic retrieval, incomplete data, matrix polynomial, polynomial rooting, rank-reduction estimator, shift-invariance property.

I. INTRODUCTION

MANY important problems in signal processing and communications naturally translate into the harmonic retrieval (HR) problem of estimating the frequencies and damping factors of a discrete harmonic mixture from one or multiple observations taken, e.g., along time, frequency or space. The undamped HR problem emerges, for example, in classical sensor array processing and direction-of-arrival (DOA) estimation applications for radar and sonar [1], [2]. Similarly, in parametric spectrum estimation [3], [4] and in parametric channel identification [5], [6], the underlying estimation problem can be formulated as an undamped HR problem. Other applications like nuclear magnetic resonance (NMR) spectroscopy [7], [8], and material damage monitoring [9], require the estimation of damped factors along with the frequencies of the superimposed harmonics. Apart from these specific applications, HR is also applicable to a fundamental class of engineering problems, in which the observed signals are modeled as auto-regressive (AR) or moving-average (MA) processes [10]. Specifically the autocorrelation function of the AR process can be represented as a damped harmonic mixture, where the harmonics corresponding to the roots of the characteristic equation are associated with the AR parameters. Similarly, the cross-correlation function associated with the MA processes leads to the undamped HR problem from which the MA parameters can be estimated.

Classical methods for the damped harmonic estimation, which include as a special case the undamped HR, require a full-search over the entire 2-D parameter space [8], [10], [11]. Pisarenko’s method [11] was among the earliest works in subspace-based HR and many modifications of this method have emerged afterwards. The state-space and the singular value decomposition approaches have been used for HR in [10]. The classical MUSIC estimator of [12], originally developed for the undamped HR, was extended to the damped HR problem in application to NMR spectroscopy in [8]. To estimate the signal parameters, this method requires an exhaustive search over its multi-dimensional spectral function and the associated computational complexity is rather high.

Search-free subspace-based methods for the undamped HR problem have gained considerable attention [2], [13]–[18]. The simple ESPRIT algorithm [1], originally developed in the context of DOA estimation, and its extensions in [2] and [13] exploit specific shift-invariances inherent in the data model. These methods can straightforwardly be generalized to the damped HR case as well. In [19] a high resolution method based on a generalization of the ESPRIT algorithm is presented which allows estimation of the damping factors. Moreover, many of these search-free methods have been extended to the two- and multi-dimensional HR case for different applications [13]–[18].

A common approach to subspace-based parameter estimation is to extract the signal- and noise-subspaces from the sample covariance matrix which can be easily estimated from the observations if multiple snapshots are available [2], [13]. If only a single snapshot is available, as is often encountered in, e.g., time-series analysis, then the rank of the estimated covariance matrix is not sufficient to span the signal subspace. In this case, forward-backward (FB) averaging [20] and smoothing techniques [10], [14], [15] can be applied to obtain an estimate of the covariance matrix with sufficient rank. In [6] and [17], a method referred to as the multidimensional folding (MDF) algorithm, was proposed for undamped HR in which the subspaces are extracted from a low-rank matrix that is directly constructed from the data samples rather than its second moments. The benefit

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of this method is that it yields the best trade-off between the subspace rank and aperture size and therefore, maximizes the number of identifiable sources for a given number of observations.

There are applications where not all sensors or samples are available, as in the cases of, e.g., sensor failure, corrupted samples, or impulsive noise. In this situation, the above methods for damped HR are not applicable or they can be applied only with a severe loss of information. This is due to the fact that these methods uses the regular structure of the data model which becomes destroyed if specific samples are missing. Due to the lack of structure, it may then not even be possible to use all the available data samples. In recent years, several methods for estimating the harmonic frequencies in the missing sample case have been introduced. The special case of the gapped data where only segments of data are available has been studied in [21] where a weighted least square (WLS) technique has been proposed. In [22], the missing-data amplitude and phase estimation (MAPES) technique are presented which estimates the signal frequencies by using the expectation maximization algorithm iteratively. Based on the iterative adaptive approach (IAA) method of [23], the missing-data IAA (MIAA) in [24] has been introduced for estimating the missing data along with the harmonic frequencies. Although both the MIAA and the MAPES method are search-based methods, yet as claimed by their authors, the MIAA is much faster than the MAPES method. While in the undamped HR, search-based techniques such as MIAA [24] and rooting-based methods such as root-MUSIC [25] naturally extend to account for missing samples, they represent a major problem in the damped HR case.

In this paper (which extends the results of [18] and [26]), we introduce the weighted multiple invariance (WMI) approach. This algorithm is based on a linear combination of the rank-reduction criteria obtained from different shift-invariants of the signal model. For the complete sample case, we derive conditions under which the damped harmonic parameters can be uniquely estimated from the roots of a matrix polynomial (MP). Then, the case of incomplete samples is studied where we prove that spurious roots that do not correspond to the true signal harmonics are generally obtained. To overcome this ambiguity, new methods are proposed. First, the idea to select the best estimate according to the MUSIC criterion is discussed. Then, we develop a method based on polynomial intersection of two distinct polynomials in which the common factors corresponding to the true signal harmonics are directly extracted.

The remainder of the paper is organized as follows. The signal model for the one-dimensional HR problem is introduced in Section II. In Section III, the WMI approach is developed and in Section IV, the complete sample scenario is discussed. In Section V, the root selection approach is presented. Then, in Section VI, we develop a novel method to estimate the harmonics in the incomplete sample case. In Section VII, simulations are presented. Section VIII concludes the paper.

Notation: The transpose and the Hermitian transpose are denoted by \((\cdot)^T\) and \((\cdot)^H\), respectively. \(E\{\cdot\}\) denotes the statistical expectation. \(R\{\cdot\}\) and \(N\{\cdot\}\) denote the range- and the null-space of a matrix, respectively. \(\text{Tr}\{\cdot\}\) and \(\det\{\cdot\}\) stand for the trace and the determinant of a matrix, respectively. \(\text{Re}\{\cdot\}\) and \(\text{Im}\{\cdot\}\) denote, respectively, the real and imaginary part of a complex number or matrix. The absolute value of a complex number is denoted by \(|\cdot|\). The conjugate of a complex value is shown by \((\cdot)^*\). The vector \(e_p\) denotes the \(p\)th column of the identity matrix \(I\). The column-rank of a matrix is expressed as \(\text{rank}\{\cdot\}\). The Kronecker and Schur-Hadamard products are denoted by \(\otimes\) and \(\circ\), respectively.

II. SIGNAL MODEL

Let us assume that the superposition of \(P\) complex exponentials is observed in the presence of additive noise. The received signal at sample \(k\) can be expressed as [14]

\[
y_k = \sum_{p=1}^{P} a_p x_p^{k-1} + z_k; \quad k = 1, \ldots, K
\]

where \(a_p \in \mathbb{C}\) satisfies \(|a_p| < 1\) and represents the \(p\)th signal generator, \(x_p\) is the \(p\)th complex random source amplitude, \(z_k\) is the complex white zero-mean sensor noise with variance \(\sigma^2\), and \(K\) is the number of samples. Note that (1) describes a generic signal model in which, depending on the application, \(y_k\) may express a time series at sample \(k\), a spatial sample taken from the \(k\)th sensor of a uniform linear array, or, more generally, a pre-processed data sample represented in a transformed domain, e.g., the \(k\)th discrete Fourier coefficient. In this paper we assume that the number of signals \(P\) is supposed to be known or can be estimated, e.g., from methods discussed in [27] or [28].

Stacking the samples in vectors and defining \(y \triangleq [y_1, y_2, \ldots, y_K]^T\), \(a_p \triangleq [a_{p,1}, a_{p,2}, \ldots, a_{p,P}]^T\), and \(z \triangleq [z_1, \ldots, z_K]^T\), respectively, to be the received signal vector, the signal generator vector corresponding to the \(p\)th signal and the zero-mean Gaussian noise vector with the covariance matrix equal to \(\sigma^2 I\), (1) can be compactly written as

\[
y = \sum_{p=1}^{P} a_p x_p + z.
\]

If \(N\) observation vectors (snapshots) are available, then we obtain the following signal model:

\[
y(n) = Hx(n) + z(n); \quad n = 1, \ldots, N
\]

where \(x(n) \triangleq [x_1(n), x_2(n), \ldots, x_P(n)]^T\) is the signal vector at snapshot index \(n\) and the \(K \times P\) signal generator matrix is defined as

\[
H \triangleq [a_1, a_2, \ldots, a_P] \\
\triangleq [h_1^T, h_2^T, \ldots, h_K^T]^T.
\]

Here, the \(1 \times P\) vector \(h_k = [a_1^{k-1}, a_2^{k-1}, \ldots, a_P^{k-1}]^T\) denotes the \(k\)th row of \(H\).

In this paper, a subspace-based harmonic retrieval method is proposed where the subspace matrices are obtained from the eigendecomposition of the sample covariance matrix. In literature, several subspace smoothing methods are available (see, e.g., [10], [14], [15] and [17]) in which the signal and noise subspaces are obtained from a single signal data snapshot, i.e., for \(N = 1\).
In the incomplete sample case, we assume that the observations are only available at specific sample locations. These samples are indicated by the binary indicator $t_k$ which takes the values

$$t_k = \begin{cases} 1, & \text{if the } k\text{th sample is available;} \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

The received signal model (3) then becomes

$$\tilde{y}(n) = \mathbf{T}_{\mathbf{y}}(n) = \mathbf{TH}\mathbf{x}(n) + \mathbf{Tz}(n) = \mathbf{H}_{\mathbf{x}}(n) + \bar{z}(n); \quad n = 1, \ldots, N \quad (6)$$

where

$$\mathbf{T} \triangleq \text{diag}\{t_1, t_2, \ldots, t_K\} \quad (7)$$

is the diagonal matrix containing the binary entries defined in (5), and $\mathbf{H} \triangleq \mathbf{TH}$ and $\bar{z}(n) \triangleq \mathbf{Tz}(n)$ denote the $K \times P$ signal generator matrix and the $K \times 1$ noise vector at snapshot $n$, respectively. Note that for notational convenience we represent the missing samples with zero entries at the respective positions using the diagonal matrix $\mathbf{T}$, rather than completely removing these entries from the data. The dimensions and indices of the vectors and matrices in the incomplete sample case are thus equivalent to their corresponding counterparts in the complete sample case. The data covariance matrix is defined as

$$\mathbf{R} \triangleq \mathbb{E}\{\tilde{y}(n)\tilde{y}^H(n)\} = \mathbf{T}\mathbb{E}\{\mathbf{y}(n)\mathbf{y}^H(n)\}\mathbf{T}^H = \mathbf{THR}_{\mathbf{x}}\mathbf{H}^H + \sigma^2\mathbf{T} = \mathbf{HR}_{\mathbf{x}}\mathbf{H}^H + \sigma^2\mathbf{T} \quad (8)$$

where $\mathbf{R}_{\mathbf{x}} = \mathbb{E}\{\mathbf{x}(n)\mathbf{x}^H(n)\}$ is the signal covariance matrix and the properties $\mathbf{T}^H = \mathbf{T}$ and $\mathbf{T}\mathbf{T}^H = \mathbf{I}$ are taken into account. The eigendecomposition of the data covariance matrix is given by

$$\mathbf{R} = \mathbf{U}_S\mathbf{A}_S\mathbf{U}_S^H + \mathbf{U}_N\mathbf{A}_N\mathbf{U}_N^H \quad (9)$$

where $\mathbf{A}_S$ and $\mathbf{A}_N$ are the diagonal matrices containing the $P$ signal- and $(K - P)$ noise-subspace eigenvalues, respectively. Similarly, $\mathbf{U}_S$ and $\mathbf{U}_N$ are the $K \times P$ and $K \times (K - P)$ matrices whose columns are the signal- and noise-subspace eigenvectors associated with the signal- and noise-subspace eigenvalues, respectively. In practice, the true covariance matrix is not available and, therefore, its sample estimate can be used

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{n=1}^{N} \tilde{y}(n)\tilde{y}^H(n) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{T}\mathbf{y}(n)\mathbf{y}^H(n)\mathbf{T}^H \quad (10)$$

The eigendecomposition of $\hat{\mathbf{R}}$ can be written as

$$\hat{\mathbf{R}} = \hat{\mathbf{U}}_S\hat{\mathbf{A}}_S\hat{\mathbf{U}}_S^H + \hat{\mathbf{U}}_N\hat{\mathbf{A}}_N\hat{\mathbf{U}}_N^H \quad (11)$$

where the diagonal matrices $\hat{\mathbf{A}}_S$ and $\hat{\mathbf{A}}_N$ contain the $P$ signal- and $(K - P)$ noise-subspace eigenvalues of $\hat{\mathbf{R}}$, respectively, and $\hat{\mathbf{U}}_S$ and $\hat{\mathbf{U}}_N$ are the $K \times P$ and $K \times (K - P)$ matrices whose columns are the signal- and noise-subspace eigenvectors of $\hat{\mathbf{R}}$.

### III. THE WMI APPROACH

Based on the Vandermonde structure of the signal generator matrix $\mathbf{H}$ defined in (4), a shift-invariance property can be established that relates the $(m + k)$th row of $\mathbf{H}$ with its respective $m$th row as

$$\mathbf{h}_{m+k} = \mathbf{h}_m\Delta^k \quad (12)$$

for $k = 1, \ldots, K - 1$, where the diagonal shifting matrix is defined as

$$\Delta \triangleq \text{diag}\{a_1, a_2, \ldots, a_P\} \quad (13)$$

These invariance (12) are utilized in the classical ESPRIT algorithm of [1] for a particular $k$. Here, we follow a similar idea as in [13] where multiple shift-invariances are used simultaneously. Note, however, that due to the uniform sampling grid, the shift-invariance (12) is valid for arbitrary integer shifts $k = 1, \ldots, K - 1$. It is, therefore, meaningful to utilize the properties of all the shifts by forming a linear combination of weighted invariance (12) for different shifts. Thus, multiplying (12) with $\mathbf{h}_m^H$ from the left, and forming a linear combination with weighting coefficients $\{c_k\}_{k=1}^K$, we obtain

$$\sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k t_{m+k} t_m (\mathbf{h}_m^H \mathbf{h}_{m+k} - \mathbf{h}_m^H \mathbf{h}_m \Delta^k) - 0_{P \times P} \quad (14)$$

Note that $t_m$ and $t_{m+k}$ in (14) reflect the fact that in the incomplete sample case only shift-invariances for which both the samples $y_m$ and $y_{m+k}$ are available can be used in (12). Let $\mathbf{e}_p$ denote the $p$th column of the identity matrix $\mathbf{I}$. Multiplying both sides of (14) from right with $\mathbf{e}_p$, we obtain

$$\sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k t_{m+k} t_m (\mathbf{h}_m^H \mathbf{h}_{m+k} - \mathbf{h}_m^H \mathbf{h}_m \Delta^k) \mathbf{e}_p = 0_{P \times 1} \quad (15)$$

where we used the property

$$\Delta^k \mathbf{e}_p = a_p^k \mathbf{e}_p \quad (16)$$

Therefore, defining the MP

$$\mathbf{M}(a) \triangleq \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k t_{m+k} t_m (\mathbf{h}_m^H \mathbf{h}_{m+k} - \mathbf{h}_m^H \mathbf{h}_m \Delta^k) \mathbf{e}_p$$

we observe from (15) that $\mathbf{M}(a) \mathbf{e}_p = 0_{P \times 1}$, which means that $\mathbf{M}(a)$ drops rank when the parameter $a$ in the above expression
becomes identical to one of the true signal generators, i.e., when \( a \) takes one of the values \( a_1, \ldots, a_P \).

From (8) and (9) it follows that the generator matrix \( \tilde{H} \) and the signal eigenvector matrix \( U_S \) span the same subspace. Thus, there exists a \( P \times P \) full-rank matrix \( K \) such that

\[
U_S = \tilde{H}K
\]  

(18)

and similarly for the rows corresponding to \( t_m = 1 \), we have

\[
u_{S,m} = t_m h_m^T K
\]  

(19)

where \( u_{S,m} \) denotes the \( m \)-th row of \( U_S \). Multiplying (17) from left and right with the full-rank matrices \( U_S \) and \( K \), respectively, and using (19), we obtain the MP

\[
G(a) = K^H M(a) K
\]

\[
= \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k t_{m+k} k t_m \cdot \left( u_{S,m+k}^T u_{S,m+k} - u_{S,m+k}^T u_{S,m+k} a^k \right)
\]  

(20)

Due to the full-rank property of the mixing matrix \( K \), the MPs in (17) and (20) are equivalent in the sense that both MPs share the same set of roots, i.e., \( G(a) \) becomes singular if and only if \( M(a) \) becomes singular.

The MP \( G(a) \) in (20) will be used in Sections IV and V to estimate the signal generators from the singularities of the MP \( G(a) \), where the estimate of the signal subspace matrix \( U_S \) will be used instead of the true signal subspace matrix \( U_S \). However, for the theoretical analysis of the singularities of the MP \( G(a) \), we will assume the ideal (noise-free) case and use the signal generator matrix \( H \) instead of the signal subspace matrix \( U_S \). Thus, in the following we investigate the singularities of the equivalent MP \( M(a) \) defined in (17).

As shown in (A.4) of Appendix A, the MP in (17) can be factorized as

\[
M(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k t_{m+k} k t_m \cdot \left( I - \Delta^{-1} a \right)
\]

\[
= \tilde{H}^H C(a) \tilde{H} (I - \Delta^{-1} a)
\]

\[
= M_t(a) M_s(a)
\]

(21)

where the \((K-1) \times P\) matrix

\[
H = [h_1^T h_2^T \ldots h_K^T]^T
\]

(22)

is formed from the last \((K-1)\) rows of the signal generator matrix \( H \) in (4), \( C(a) \) is defined in (23), shown at the bottom of the page, and

\[
M_s(a) \triangleq (I - \Delta^{-1} a)
\]

\[
M_t(a) \triangleq \tilde{H}^H C(a) \tilde{H}
\]

(24)

(25)

are the signal and residual MPs, respectively.

From the factorization in the last row of (21), we observe that \( M(a) \) becomes singular if \( a \) is equal to one of the true signal generators \( a_1, \ldots, a_P \). This stems from the fact that in this case one of the diagonal entries of \( M_s(a) \) becomes zero. In this case, all entries in the corresponding column of \( M(a) \) are zeros which leads to a singular MP. Hence, we conclude, that the singularity of \( M(a) \) is necessary for \( a \) to become equal to one of the true signal generators \( a_1, \ldots, a_P \). Due to the full-rank property of the mixing matrix \( K \), the similar statement holds true for the MP \( G(a) \) as well. Next, let us investigate under which conditions the singularity of \( M(a) \) and \( G(a) \) is also sufficient for \( a \) to be equivalent to one of the true generators. This question is of particular importance as the true generators will be later obtained from the singularities of the respective MPs. In this case, the sufficient conditions on \( M(a) \) and, subsequently, \( G(a) \) directly reveal conditions under which unique estimates can be obtained from the roots of the MPs.

To answer this question, consider the first factor in the factorization of (21), i.e., the residual matrix polynomial \( M_t(a) \) defined in (25). Clearly, if \( M_t(a) \) is non-singular for \( a \) inside or on the unit-circle, then the MPs \( M(a) \) and \( G(a) \) become singular if and only if \( a \) corresponds to one of the true signal generators \( a_1, \ldots, a_P \). In the following section, we derive, for the complete sample case, simple conditions on the weighting coefficients \( \{ c_k \}_{k=1}^{K-1} \) for which \( M_s(a) \) does not display spurious roots that do not correspond to the true signal generators inside or on the unit-circle. We show that in this case, unique estimates can be found from the roots of \( G(a) \) (or \( M(a) \)) obtained inside or on the unit-circle. In Section V, we consider the incomplete sample case. In the latter section, we show that general uniqueness conditions, i.e., the conditions that do not depend on the data, cannot be established and propose an estimation procedure from which unique parameter estimates can be obtained.

IV. THE COMPLETE SAMPLE CASE

In this section, we consider the case when all the data samples are available. Hence, we assume that \( t_k = 1 \) for \( k = 1, \ldots, K \). In this case, the MPs in (17) and (20) reduce to

\[
M_{cs}(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k t_{m+k} h_{m+k}^H h_{m+k} (I - \Delta^{-k} a^k)
\]

(26)

is given by

\[
C(a) \triangleq \begin{bmatrix}
t_2 t_1 c_1 & 0 & \cdots & 0 \\
t_3 t_1 c_2 & t_3 (t_1 c_2 + t_2 c_1) & \cdots & 0 \\
t_4 t_1 c_3 a^2 & t_4 (t_1 c_3 + t_2 c_2) a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
t_K t_1 c_K - 1 a^{K-2} & t_K \left( \sum_{i=1}^{K-1} t_i c_{K-i} \right) a^{K-3} & \cdots & t_K \left( \sum_{i=1}^{K-1} t_i c_{K-i} \right) a^{K-4} \\
\end{bmatrix}
\]

(23)
and
\[
G_{cs}(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k \left( u_{S,m+k}^H u_{S,m+k} - u_{S,m+k}^H \hat{u}_{S,m+k} a^k \right)
\]  \hspace{1cm} (27)

where the subscript “cs” indicates the complete sample case. The following theorem holds.

**Theorem:** Provided that \( \mathbf{H} \) in (22) is full column-rank and that the weighting coefficients in (26) and (27) satisfy
\[
2c_1 \geq c_2 \geq \cdots \geq c_{K-1} \geq 0; \quad c_1 > 0 \quad (28)
\]
the \( P \times P \) matrices \( \mathbf{M}_{cs}(a) \) and \( \mathbf{G}_{cs}(a) \), evaluated inside and on the unit-circle, i.e., for \( |a| \leq 1 \), become singular if and only if \( a = a_p \) for \( p = 1, \ldots, P \).

**Proof:** As discussed in the previous section, it is straightforward to see from the factorization in (21), that the MP \( \mathbf{M}_{cs}(a) \) is singular if \( a \) corresponds to a true generator. In order to prove that with the choice of (28) the singularity of \( \mathbf{M}_{cs}(a) \) inside the unit-circle is also a sufficient condition for \( a \) to correspond to a true generator, we show that the residual polynomial \( \mathbf{M}_{r}(a) \), when \( t_k = 1 \) for \( k = 1, \ldots, K \), is always non-singular. Making use of Sylvester’s rank inequality [30] we have

\[
\text{rank}\{ \mathbf{H} \} + \text{rank}\{ \mathbf{C}(a) \} \leq K + 1
\]
\[
\leq \text{rank}\{ \mathbf{H}^H \mathbf{C}(a) \mathbf{H} \}
\]
\[
\leq \min\{ \text{rank}\{ \mathbf{C}(a) \}, \text{rank}\{ \mathbf{H} \} \}. \quad (29)
\]

Considering that \( \mathbf{H} \) is of full column-rank \( P \leq K - 1 \), the matrix \( \mathbf{M}_{r}(a) = \mathbf{H}^H \mathbf{C}(a) \mathbf{H} \) is of full column- and row-rank \( P \) if \( \mathbf{C}(a) \) is non-singular. It remains to be shown that under the conditions in (28), the MP \( \mathbf{C}(a) \) does not possess any singularities inside (or on) the unit-circle. Our proof of the latter fact can be found in Appendix B.

Note that the Vandermonde matrix \( \mathbf{H} \) is full-rank if and only if all generators \( a_1, \ldots, a_P \) are distinct and if the number of damped harmonics is less than the number of samples \( K \) [30]. Choosing the weights \( \{ c_k \}^{K-1}_{k=1} \) according to (28) in the complete sample case, the true generators \( a_1, \ldots, a_P \) can thus be uniquely obtained from the roots of (27).

In practice, in order to root the MPs in (17) and (20), two different approaches can be commonly applied. In the first approach the determinant of the MP is recursively calculated using cofactor expansion and then the roots of the acquired scalar polynomial are determined. To obtain the coefficients of the determinant polynomial, the fast Fourier transform (FFT) and the inverse fast Fourier transform (IFFT) can be used where convolution in the polynomial coefficient domain is replaced by IFFT, multiplication and FFT [29]–[31]. In the second approach the roots of the MP are computed from the \( P \) smallest generalized eigenvalues of the associated block companion matrix pair [30], [32].

In the finite sample case, when only estimates of the rows of the signal subspace matrix \( \mathbf{U}_S \) in (27) are available, we propose the following procedure to estimate the signal generators:

**Algorithm I**

1. Compute the sample covariance matrix \( \hat{\mathbf{R}} \) in (10) and find the signal eigenvectors matrix \( \mathbf{U}_S \).
2. Obtain the sample version of the matrix polynomial in (27):
\[
\hat{\mathbf{G}}_{cs}(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} c_k \left( \mathbf{u}_{S,m+k}^H \hat{\mathbf{u}}_{S,m+k} - \mathbf{u}_{S,m+k}^H \hat{\mathbf{u}}_{S,m+k} a^k \right). \quad (30)
\]
3. Calculate the \( P \) smallest magnitude roots \( \hat{a}_1, \ldots, \hat{a}_P \) of \( \hat{\mathbf{G}}_{cs}(a) \), e.g., from the \( P \) smallest generalized eigenvalues of the block companion matrix pair associated with \( \hat{\mathbf{G}}_{cs}(a) \) [29], [30], [32].

**IV. THE INCOMPLETE SAMPLE CASE**

In the incomplete sample case, we consider the general MPs \( \mathbf{M}(a) \) and \( \mathbf{G}(a) \) as defined in (17) and (20), respectively. Similar to the complete sample case, we investigate the singularities of the residual MP in the factorization (21). Note, however, that the residual MP \( \mathbf{M}_r(a) \) in (25) cannot be computed analytically from the available data samples, simply because not all the rows of complete-data (row-reduced) signal matrix \( \mathbf{H} \) in (25) are observable.

According to (21), \( \mathbf{M}(a) \) can be expressed as
\[
\mathbf{M}(a) = \mathbf{H}^H \mathbf{C}(a) \mathbf{H} (\mathbf{I} - \Delta^{-1} a^1). \quad (31)
\]
where \( \mathbf{C}(a) \) is defined in (23). In contrast to the complete sample case, for which simple uniqueness conditions (28) are proven, similar data-independent uniqueness conditions cannot be obtained for the incomplete sample scenario. Due to the unavailability of the \( (k+1) \)th sample in the case \( t_{k+1} = 0 \), the masking operation in (23) sets all entries in the \( k \)th row of \( \mathbf{C}(a) \) in (31) equal to zero. This results in a reduced column-rank of \( \mathbf{C}(a) \). We observe from Sylvester’s rank inequality (29) that in this case the product \( \mathbf{H}^H \mathbf{C}(a) \mathbf{H} \) can become rank deficient if the null-space of \( \mathbf{C}(a) \) and the range-space of \( \mathbf{H} \) intersect. Hence, the residual matrix polynomial exhibits spurious roots at locations that depend on the choice of the weighting coefficients \( \{ c_k \}^{K-1}_{k=1} \) and on the signal generator matrix \( \mathbf{H} \), i.e., on the set of the true signal generators themselves. Thus, unlike in the complete sample case where \( \mathbf{M}(a) \) is rank-deficient if and only if \( a \) becomes equal to one of the true generators, in the incomplete sample case \( \mathbf{M}(a) \) may exhibit additional singularities which do not correspond to the true signal generators. In this case, data-independent uniqueness conditions cannot be formulated.

One way to deal with the resulting ambiguities is to develop an estimation scheme which consists of rooting the MP \( \mathbf{G}(a) \) in (20) and then separating the signal roots from the spurious roots according to some appropriate criterion. In principle, many different root-selection criteria may be applied. The most powerful but also most complex schemes involve multi-dimensional parameter subset testing, i.e., for all possible sets of \( P \) roots taken from the set of signal and spurious roots, the set that yields the
largest criterion function value is selected. An excellent criterion function may be the multi-dimensional likelihood function. The main disadvantage associated with the ML criterion, however, stems from the fact that for a large number of signal and spurious solutions, the computational complexity of the proposed scheme grows dramatically as the number of subsets of size $P$ increases exponentially.

The spectral MUSIC criterion, originally developed in the context of DOA estimation, i.e., for the undamped HR problem with $|a| = 1$, can also be used as a one-dimensional criterion to determine the validity of a root in the damped harmonic case. The following algorithm can be applied to obtain the unique signal generator estimates.

**Algorithm II**

**Step 1:** Compute the MP

$$
\hat{G}(a) = \sum_{m=1}^{K-m} \sum_{k=1}^{K-m} c_k S_{m+k} f_m + \hat{u}_S^H S_{m+k} S_{m+k} S_m u^k
$$

**Step 2:** Compute the roots $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_M \ (M \geq P)$ of $\hat{G}(a)$ inside the unit-circle.

**Step 3:** Form the Vandermonde vector $\hat{a}_m = [1, \hat{a}_m, \hat{a}_m^2, \ldots, \hat{a}_m^{K-1}]^T$ for each root.

**Step 4:** For each root, calculate $f_{\text{MUSIC}}(\hat{a}_m) = \hat{a}_m^H \hat{U} \hat{U}^H S_m u^k / (\hat{a}_m^H \hat{a}_m)$.

**Step 5:** Find the $P$ smallest minima of $f_{\text{MUSIC}}$ and their corresponding roots.

Our computer simulations show that the selection scheme based on the MUSIC criterion in Steps 4 and 5 of Algorithm II performs reasonably well and its performance achieves the performance of the benchmark (but impractical) ideal selection scheme where the estimates closest to the true generators are chosen.

It should be noted that the procedure of estimating the signal generators along with selecting the signal roots is suboptimal in the sense that due to numerical instability of the rooting procedure, the spurious roots located closely to the signal roots may adversely affect the estimates of the signal roots, resulting in a significant performance degradation at low and medium SNR values or sample sizes. Therefore, we propose in the next section a more sophisticated approach based on rank-reduction criterion to uniquely obtain the signal roots without the need of further selection.

**VI. EIGENVECTOR METHOD BASED ON POLYNOMIAL INTERSECTION**

From (31), we can observe that the signal roots of $M(a)$ remain unchanged for different sets of $\{c_k\}_{k=1}^{K-1}$. This is due to the fact that the weighting coefficients affect only the MP $M(a)$. Hence, if we replace the taken set of coefficients by any different set, only the residual roots change. Therefore, we can claim that for any two polynomials resulting from (31) with different sets of weighting coefficients, say $\{c_k\}_{k=1}^{K-1}$ and $\{c'_k\}_{k=1}^{K-1}$, there exist at least $P$ common roots. Reversely, if there are exactly $P$ common roots, then these are the signal roots.

It is well known that even small perturbations in the polynomial coefficients of MPs strongly affect the root locations. Different sets of weighting coefficients $\{c_k\}_{k=1}^{K-1}$ result in substantially different coefficients of the polynomial coefficients of $M(a)$ and $M(a)$ and, consequently, in different root locations. This has already been addressed earlier from a slightly different perspective in the context of (25), where it was mentioned that singularities of $M(a)$ may result from the intersection of the subspaces of $C(a)$ and $H$, and thus depend on both the true signal generators and the weighting coefficients $\{c_k\}_{k=1}^{K-1}$. Interestingly, this only applies to the spurious roots, i.e., the roots of $M(a)$ and, therefore, to all but the signal roots in $M(a)$, as can be seen from the factorization in (21). Therefore, as will be also confirmed by our simulations, it is unlikely for different random sets of coefficients that the residual polynomials yield exactly identical spurious roots. In order to extract the common polynomial factor of two scalar polynomials with common roots, the properties of the Sylvester matrix will be used. Before deriving the procedure, let us introduce the Sylvester matrix and some of its important properties.

**A. Sylvester Matrix and Its Properties**

Consider two scalar polynomials

$$f(a) = p_n a^n + p_{n-1} a^{n-1} + \cdots + p_1 a + p_0,$$

$$g(a) = q_m a^m + q_{m-1} a^{m-1} + \cdots + q_1 a + q_0,$$

degree $n$ and $m$, respectively, where $p_n \neq 0$ and $q_m \neq 0$. Then, the $(m + n) \times (m + n)$ Sylvester matrix can be defined as

$$S(f, g) \triangleq \begin{pmatrix}
 p_0 & p_1 & \cdots & p_n & 0 & \cdots & 0 \\
 0 & p_0 & p_1 & \cdots & p_n & 0 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & p_0 & p_1 & \cdots & p_n \\
 q_0 & q_1 & \cdots & q_m & 0 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & q_0 & q_1 & \cdots & q_m \\
 \end{pmatrix}.$$  

The resultant of $f(a)$ and $g(a)$ is defined as the determinant of the Sylvester matrix $[33]–[35]$ and is denoted as $\text{Res}(f, g) = \det(S(f, g))$. Note, that the resultant can also be written as $\text{Res}(f, g) = \prod_{i=1}^{m+n} (\alpha_i - \beta_j)$ where $\alpha_i$ and $\beta_j$ are the roots of the polynomials $f(a)$ and $g(a)$, respectively. It can be shown that the polynomials $f(a)$ and $g(a)$ have at least one common root $\hat{a}$ if and only if $\text{Res}(f, g) = 0$ $[33]–[35]$.

Furthermore, it can be seen that if $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_P$ are the common roots of $f(a)$ and $g(a)$, then the Vandermonde vectors

$$\hat{a}_p = [1, \hat{a}_p, \hat{a}_p^2, \ldots, \hat{a}_p^{m+n-1}]^T$$

for $p = 1, \ldots, P$ span the null-space of $S(f, g)$, i.e.,

$$\mathcal{N}(S(f, g)) = \mathcal{R}$$
where $\mathbf{A}$ is the matrix containing the root Vandermonde vectors in (36):

$$
\mathbf{A} = [\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_P].
$$

Let $\mathbf{V}_0$ denote the unitary matrix consisting of the eigenvectors $v_{0,p}$ associated with the zero eigenvalues of $\mathbf{S}(f, g)$ such that

$$
\mathbf{V}_0 = [v_{0,1}, v_{0,2}, \ldots, v_{0,P}].
$$

Based on these properties, the following method is proposed.

### B. The Proposed Algorithm

Consider two MPs formed from different sets of weighting coefficients $\{c_k\}_{k=1}^{K-1}$ and $\{c'_{k}\}_{k=1}^{K-1}$ according to (20)

$$
\mathbf{G}_1(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-1} c_k t_{m+k} t_m
\cdot (t^H_{m+k} \mathbf{u}_s m + k - t^H_{m+k} \mathbf{u}_s m a_k) \quad (40)
$$

and

$$
\mathbf{G}_2(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-1} c'_{k} t_{m+k} t_m
\cdot (t^H_{m+k} \mathbf{u}_s m + k - t^H_{m+k} \mathbf{u}_s m a_k). \quad (41)
$$

Assume that the Sylvester matrix $\mathbf{S}(f, g)$ is obtained from the scalar polynomials $f(a) = \det(\mathbf{G}_1(a))$ and $g(a) = \det(\mathbf{G}_2(a))$, each of the maximum degree of $2K - 2$. Note that from (20) and the full-rank property of matrix $\mathbf{K}$, it is clear that the roots of the determinants of the above MPs are equal to that of their corresponding counterparts in (31). Then, taking into account that the common roots of $f(a)$ and $g(a)$ correspond to the true signal generators $a_1, \ldots, a_P$, we note that the Sylvester matrix $\mathbf{S}(f, g)$ has its null-space of dimension $P$. Furthermore, from (37), for signal generators $a_1, \ldots, a_P$, the Vandermonde matrix $\mathbf{A}$ in (36) which corresponds to the extended version of $\mathbf{H}$, spans the null-space of $\mathbf{S}(f, g)$. Hence, with $\mathbf{V}_0$ defined in (39), we have $\mathbf{R}(\mathbf{V}_0) = \mathbf{R}(\mathbf{A})$. Interestingly, we have thus obtained the matrix $\mathbf{V}_0$ that spans the space of the $(m + n) \times P$ virtual complete data signal matrix $\mathbf{A}$. Due to the Vandermonde structure of $\mathbf{A}$, the ESPRIT approach [1] can be applied to estimate $a_1, \ldots, a_P$ from $\mathbf{V}_0$. The estimation procedure for the proposed method is described below.

### Algorithm III

**Step 1:** Compute the sample versions of the two MPs $\mathbf{G}_1(a)$ and $\mathbf{G}_2(a)$ in (40) and (41), with two different sets of $\{c_k\}_{k=1}^{K-1}$ and $\{c'_{k}\}_{k=1}^{K-1}$.

**Step 2:** Form the associated Sylvester matrix.

**Step 3:** Find the $P$ minor eigenvectors of the Sylvester matrix $\mathbf{V}_{0,p}$, $p = 1, \ldots, P$, then form $\mathbf{V}_0$.

**Step 4:** Compute the eigenvalues $\tilde{\lambda}_p$, $p = 1, \ldots, P$ of $\Phi = (\mathbf{v}_c^H \mathbf{V}_0)^{-1} \mathbf{V}_0^H \mathbf{Y}_0$, where $\mathbf{Y}_0$ and $\mathbf{V}_0$ are formed from the matrix $\mathbf{V}_0$ by removing the first and the last rows, respectively.

In Algorithm III, $\mathbf{\tilde{V}}_0$ is defined as the sample counterpart of the matrix $\mathbf{V}_0$ obtained from the sample versions of the respective MPs in (40) and (41), $\mathbf{G}_1(a)$ and $\mathbf{G}_2(a)$, where $\mathbf{V}_0 = [\mathbf{v}_{0,1}, \mathbf{v}_{0,2}, \ldots, \mathbf{v}_{0,P}]^T$ and $\mathbf{v}_{0,p}$, $p = 1, \ldots, P$ are the $P$ minor eigenvectors of the Sylvester matrix $\mathbf{S}(\mathbf{f}, \mathbf{g})$ composed of the scalar polynomials

$$
\tilde{f}(a) = \det(\mathbf{G}_1(a)) \quad (42)
$$

and

$$
\tilde{g}(a) = \det(\mathbf{G}_2(a)). \quad (43)
$$

It is worth mentioning that, in the undamped case, there is a very elegant and computationally efficient way to generate the second polynomial. The idea is to choose the coefficients of the second polynomial $\mathbf{G}_2(a)$ in (41) from the coefficients of the first polynomial, i.e., $\mathbf{G}_2(a)$ in (40), such that $c'_{k} = c^*_{k+1}$ for $k = 1, \ldots, K-1$. Then it can readily be verified that the roots of $\mathbf{G}_2(a)$ become the conjugate reciprocal of the roots of $\mathbf{G}_1(a)$, i.e., if $a_k$ is a root of $\mathbf{G}_1(a)$ then $1/a_k^*$ is a root of $\mathbf{G}_2(a)$. In this case any root on the unit-circle $(|a| = 1)$ becomes a common root of both polynomials $\mathbf{G}_1(a)$ and $\mathbf{G}_2(a)$. For each root of $\mathbf{G}_1(a)$ outside (or inside) the unit-circle, however, there exists a corresponding root of $\mathbf{G}_2(a)$ inside (or outside) the unit-circle. Hence, in general only the (signal) roots on the unit-circle are common roots of both polynomials. Therefore, applying the polynomial intersection method (Algorithm III) to these two polynomials, the signal generators can uniquely be obtained.

In the simulation section, the weights $\{c_k\}_{k=1}^{K-1}$ are chosen to be either all one or randomly decreasing. To avoid any possible loss in the performance, both threshold-wise and asymptotically, due to the arbitrary choice of the weights, we propose the following iterative refinement procedure.

### C. Adaptive Selection of Weighting Coefficients

Until now, the proper choice of the sets of weighting coefficients $\{c_k\}_{k=1}^{K-1}$ and $\{c'_{k}\}_{k=1}^{K-1}$ to form the two matrix polynomials $\mathbf{G}_1(a)$ and $\mathbf{G}_2(a)$ has not been specified. Clearly, if some specific roots of the residual polynomials become identical, i.e., for a particular choice of $\{c_k\}_{k=1}^{K-1}$ and $\{c'_k\}_{k=1}^{K-1}$ the true signal roots are no longer the unique common roots of $f(a)$ and $g(a)$, then the rank of the null-space of $\mathbf{S}(f, g)$ is higher than $P$. In this case, the proposed HR estimation procedure does not yield unique solutions. As discussed above, the event that the residual polynomials share exactly identical roots is very unlikely for randomly generated coefficient sets. In practice, however, for given sets of coefficients $\{c_k\}_{k=1}^{K-1}$ and $\{c'_k\}_{k=1}^{K-1}$, some particular roots of the corresponding residual polynomials can become close to each other. This can cause performance degradation of the proposed algorithm due to the fact that these spurious roots are associated with very small eigenvalues of the $\mathbf{S}(f, g)$. This may lead to the so-called subspace swap effect in Step 3 of Algorithm III. In particular, due to finite sample size, the nominal $(P + 1)$th minor eigenvalue of $\mathbf{S}(f, g)$ may become larger than the nominal $P$th minor eigenvalue. This effect becomes especially pronounced in the case of low SNRs, where the polynomial coefficients of $f(a)$ and $g(a)$ both may have substantial deviations from their ideal positions. Therefore, we propose to check the ratio of the $P$th and $(P + 1)$th smallest eigenvalues before forming the matrix $\mathbf{V}_c$. If this ratio is close
to one, then new random weights have to be generated and the algorithm must be repeated. These iterations continue until either the eigenvalue ratio becomes larger than some pre-defined threshold or the number of iterations exceeds a certain pre-determined critical value.

D. Computational Complexity

The computational complexity of our proposed Algorithm I, II, and III is mainly determined by the polynomial rooting procedure. In general the computational complexity required for rooting a scalar polynomial of degree $M$ corresponds to complexity required for computing the eigenvalues of the associated companion matrix. Hence the complexity is in the order of $O(M^3)$ complex operations. However, due to the highly-sparse structure of this matrix and accounting for the fact that only the $P$ minor eigenvalues are required to be computed, depending on the implementation, in practice, the computational complexity can be much lower than the nominal order. Let us assume that a largest shift of $K_0 (\leq K - 1)$ is used in our algorithms. Therefore, the degree of the polynomials in (42) and (43) becomes equal to $PK_0$. Thus, ignoring potential computational savings due to sparsity, for Algorithms I and II, since only one of these polynomials is used the complexity is in the order of $O(P^3K_0^3)$ operations. For Algorithm III, due to the eigendecomposition of the $2PK_0 \times 2PK_0$ sparse Sylvester matrix, the complexity is in the order of $O(8P^3K_0^3)$ operations.

VII. SIMULATIONS

In this section, we numerically compare the performance of the proposed methods in different scenarios. In the case of undamped harmonics, the state-of-the-art MIAA method of [24] is also included in the figures. In this search-based approach that is only applicable to undamped HR, the frequencies are estimated from a single snapshot. However, MIAA allows a natural extension to the case of multiple snapshots which we have used in this paper for a fair comparison. In this extension the estimated power at each frequency grid point is computed from each snapshot and then averaged. Note that in the simulations the noise is generated from a Gaussian distribution.

In the first example, the complete sample case is addressed. Four cases of choosing the weighting coefficients are considered: the case where $c_1 = 1$ and the remaining coefficients are zero (in this case, our algorithm reduces to the conventional ESPRIT algorithm), the case of uniform weighting coefficients where $c_k = 1$ for $k = 1, \ldots, K - 1$, the case of linearly decreasing weighting coefficients, and the case where new decreasing random set of weights in each Monte-Carlo run is used. The sample support is $K = 8$ and the number of snapshots is $N = 50$. We consider $P = 2$ damped harmonics with $a_1 = 0.6777 - 0.4464j, a_2 = 0.8442 + 0.4508j$. In all simulations, we estimate the harmonics parameters and display the root-mean-square errors (RMSE) of the estimates $\hat{a}_1, \hat{a}_2$ for different values of SNRs. Here, SNR is defined as

$$\text{SNR} \triangleq 10 \log \left[ \frac{\text{Tr}(HRH^H)}{\sigma^2KP} \right].$$  \hspace{1cm} (44)

This definition is consistent with the definition of the SNR in the undamped harmonic case, which is commonly used in array processing applications. For each of the SNR values, 500 Monte-Carlo runs were performed.

The RMSE performance of the harmonic estimation obtained from the proposed methods for different weighting coefficients is displayed in Fig. 1. These results are compared to the stochastic Cramér-Rao bound (CRB) that is derived in Appendix C. We observe a gain of about 2 dB in the case of linearly decreasing or uniform weighting coefficients (where all the available shifts are used) as compared to the case of conventional ESPRIT with the single non-zero weight. We observe that the performance of our proposed Algorithm I is robust with respect to the choice of the coefficients. This robustness can further be observed from Fig. 2 where the histogram of the RMSE for 10000 random sets of weighting coefficients and SNR $= -1$ dB.

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As the second simulation example, we consider the undamped HR in the complete sample case. In Fig. 3, we compare the RMSE of Algorithm I with different sets of coefficients and the MIAA method for the undamped case versus SNR. It has been observed that in the case of small sample size $K$ or when the parameters (i.e., the frequencies) are closely separated the MIAA is incapable to resolve the parameters even at high SNRs. Therefore, in order to obtain a meaningful comparison where all sources are resolved the MIAA requires a sufficiently large sample size. Our methods, on the other hand, offer high resolution estimation performance. In this example, we assume $K = 30$, $N = 20$, two undamped signals with $a_1 = 0.8660 + 0.5000j$ and $a_2 = 0.8117 + 0.5840j$, and 100 Monte-Carlo runs at each SNR. For the MIAA, the search over the radial frequency $\omega = \arg(a) \in (0, 2\pi)$ is performed over $10K$ grid points and at most ten iterations are used. Our proposed method shows significantly better performance than the MIAA in this scenario and runs approximately 30 times faster than the MIAA. In specifc this example, the average run-time for the MIAA, root-MUSIC, and Algorithm I are, respectively, 0.8906, 0.0125, and 0.0325 seconds. We observe that the performance of the MIAA improves significantly when we increase $K$ to 100 or more at the cost of substantially increased amount of computational time (figures are omitted due to space limitations). Another drawback of the MIAA is its limited accuracy as the resolution capability of the MIAA is very much dependent on the number of search grid points. This means that in order to stay close to the CRB the MIAA requires not only a large amount of samples but also more grid points which increases its computational complexity even further.

In the third example, we consider the incomplete sample case. We assume three damped harmonics with $a_1 = 0.8636 + 0.4677j$, $a_2 = 0.6799 + 0.5932j$ and $a_3 = 0.6678 - 0.4511j$. The sample support is $K = 10$ and only 5 effective samples are available, i.e., $t_k = 1$ for $k = 1, 3, 4, 6, 10$ and $t_k = 0$ for $2, 5, 7, 8, 9$. The number of snapshots is $N = 20$. In all simulation results, the weighting coefficients $c_k$ of the first set, $\{c_k\}_{k=1}^{K-1}$, are all chosen to be equal to one. In this case, as also confirmed by simulations, the generator can no longer be determined uniquely from Algorithm I. Thus, Algorithm II or Algorithm III must be applied to obtain unique estimates of the signal generator parameters. Therefore, the second set of weighting coefficients $\{c_k\}_{k=1}^{K-1}$ is chosen from a uniform random distribution on the interval $[0, 2]$. In Fig. 4, the performance of the different methods is displayed and compared with the corresponding CRB. From Fig. 4, it can be observed that in this scenario, Algorithm II can successfully distinguish between the true signal roots and the spurious signal roots. Moreover, our extra simulations (not included here) show that the selection scheme based on the MUSIC criterion performs as good as the idealistic selection scheme based on the knowledge of the true generators. Despite this, the eigenvector method based on polynomial intersection still outperforms the selection-based method.

In Fig. 5, we display the resolution probability of the two estimated signals with the closest signal generators, which in
our scenario are given by $a_1$ and $a_2$. We assume that these two signals are resolved when

$$\sum_{p=1}^{2} |\tilde{a}_p - a_p| \leq |a_1 - a_2|.$$  \hspace{1cm} (45)

This criterion is a modification of the one in [36] and [37] used for resolution probability of the estimated DOA.

To demonstrate the effect of the adaptive selection of the weighting coefficients in Algorithm III, the performance for different number of iterations in the same scenario is displayed in Fig. 6. It can be observed that after approximately 10–20 iterations no substantial performance improvement can be seen. From Fig. 6 we can see that the choice of the weighting coefficients can improve the threshold performance, however, the choice has little effect on the asymptotic performance of the algorithm.

In the fourth example, in Fig. 7, we consider the undamped case ($|a| = 1$) where $a_1 = 0.5582 - 0.8297j$ and $a_2 = 0.4475 - 0.8943j$. This case corresponds to the DOA estimation problem with a uniform linear array of ten sensors and two sources at the angles $\theta_1 = 18.15^\circ$ and $\theta_2 = 26.63^\circ$ relative to the broadside of the array. Therefore, we compare our results also to the results obtained from the root-MUSIC algorithm for the case where the corresponding sensors are missing. As before, we choose $t_i = 1$ for $i = 1, 3, 4, 6, 10$ and $N = 50$. As illustrated by the figure, the eigenvector method based on polynomial intersection demonstrates better performance than the selection scheme and the root-MUSIC technique. Moreover, it can be observed from Fig. 8 that the proposed method outperforms the selection scheme and the root-MUSIC method in terms of source resolution probability. It should be noted that the MIAA is unable to resolve the signal generators in this scenario. However, we observe that in this scenario the MIAA using 30 iterations and different number of uniform grid points, e.g., 100, 1000, and 10000 points, is only able to resolve the signals around $\text{SNR} = 45$ dB, irrespectively of the number of grid points used.

As the last example, in Fig. 9, we assume two undamped signals with the generators $a_1 = 0.5143 - 0.8576j$ and $a_2 = 0.2163 - 0.9776j$. The number of samples is increased to $K = 20$, the number of snapshots is $N = 30$, and $t_k = 1$ for $k = 1, 4, 8, 9, 10, 12, 17, 19, 20$ and $t_k = 0$ otherwise. As mentioned in the second example, the number of samples is increased such that the sources can be resolved using the MIAA. For such a large sample size $K$ the degree of the polynomials in (42) and (43) becomes very large, leading to large computational complexity. However, in this case we can reduce the degree of the polynomials in (42) and (43) by limiting the maximum shift, i.e., confining the index $k$ in (40) and (41) to a value $K_0$ smaller than $K - 1$. In other words, $c_k = 0$ for $k = K_0 + 1, \ldots, K$. In this example, we assume $K_0 = 9$. For the MIAA, the frequency search is performed over $10K$ grid points and at most ten iterations are used. The simulations revealed that our method performs more than 25 times faster than the MIAA. In specific in this example, the average run-time for the MIAA, root-MUSIC, Algorithm
Fig. 9. Performance comparison of different methods for the undamped HR incomplete sample case ($|\alpha| = 1$).

II, and Algorithm III are, respectively, 0.4746, 0.0051, 0.0083, and 0.0182 seconds. The limited RMSE accuracy of the MIAA is apparent in this figure and can be explained by the limited search grid resolution. In this figure, we also display the performance of our proposed method using the conjugate and backward re-ordering of the first polynomial coefficients which has been explained at the end of Section VI.

VIII. CONCLUSION

The damped HR problem has been studied. A novel weighted multiple invariance approach for harmonic estimation has been proposed that is based on polynomial rooting. Furthermore, the uniqueness of the proposed estimator has been studied. For the complete sample case, simple conditions on the weighting coefficients have been derived which guarantee unique estimation of the true signal generators. However, in the incomplete sample case, we show that such data-independent uniqueness conditions for the estimates cannot be obtained in general. Two extensions of the proposed rooting technique have been developed which yield unique estimates of the true signal generators. In the first method, the signal roots are selected based on the MUSIC criterion, whereas in the second method, the signal roots are directly estimated without any need of such selection step. The latter method relies on polynomial intersection of two distinct polynomials with common signal roots. Simulation results have demonstrated that in the complete sample case, our method outperforms the conventional single-invariance ESPRIT technique. In the incomplete sample case, simulations confirm that the proposed algorithms yield highly accurate and unique estimates of the true signal generators.

APPENDIX A

PROOF OF (21)

According to Section III, we can obtain the following multiple-shift (17):

$$M(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} \sum_{n=0}^{K-m-k} \mathbf{c}_k \mathbf{H}_{m+k,n} \mathbf{h}_{m+k} \mathbf{H}_{m+k,n} \mathbf{I} - \Delta^{-k} a^k$$

for $a = a_r$ and $c_k \in \mathbb{R}^+$, respectively. Defining $\mathbf{M}_m(a) \triangleq (\mathbf{I} - \Delta^{-1} a^1)$, it can be readily verified that $\mathbf{M}(a)$ in (A.1) can be factorized as

$$\mathbf{M}(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} \sum_{n=0}^{K-m-k} \mathbf{c}_k \mathbf{H}_{m+k,n} \mathbf{I} - \Delta^{-k} a^k \mathbf{M}_m(a)$$

Furthermore

$$\mathbf{h}_k^H = \mathbf{H}_k^H \mathbf{e}_k = \mathbf{H}_k^H \mathbf{e}_{k-1}$$

where the $(K-1) \times P$ matrix $\mathbf{H}_k$ is the matrix $\mathbf{H}$ with the dropped first row. Then, (A.2) translates to

$$\mathbf{M}(a) = \sum_{m=1}^{K-1} \sum_{k=1}^{K-m} \sum_{n=0}^{K-m-k} \mathbf{c}_k \mathbf{H}_{m+k,n} \mathbf{I} - \Delta^{-k} a^k \mathbf{M}_m(a)$$

where $E_{k,l}$ denotes the $(K-1) \times (K-1)$ matrix containing a one in the $(k, l)$ entry and zeros elsewhere, and

$$C_m(a) = \sum_{m=1}^{K-1} C_m(a).$$

Here, $C_m(a)$ is defined as

$$C_m(a) = t_m \sum_{k=1}^{K-m} \sum_{n=0}^{K-m-k} E_{m+k-1,m+k-n-1} a^n$$

for $m = 1, \ldots, K - 1$. Hence, the matrix $C(a)$ can be represented more explicitly using (23).
In Appendix A, it has been demonstrated that for the complete sample case, the MP $M(a)$ in (26) can be factorized as

$$M(a) = H^H C(a)|_{t_i=1} H(I - \Delta^{-1} a)$$  \hspace{1cm} (B.1)

where the matrix $C(a)$ in (23) transforms to (B.2), at the bottom of the page. The second term in the right-hand side (RHS) of (B.1), i.e., $(I - \Delta^{-1} a)$, becomes singular if and only if $a$ is equal to one of the true generators $a_1, \ldots, a_J$. From the definition in (25), we conclude that for the complete sample case

$$M_c(a) = H^H C(a)|_{t_i=1} H$$  \hspace{1cm} (B.3)

Recall that, in order to prove that under condition (28) all singularities of $M(a)$ inside the unit-circle correspond to true signal generators, it is sufficient to show that $M_c(a)$ is positive definite and, hence, non-singular for $|a| \leq 1$. The non-Hermitian matrix $M_c(a)$ is referred to as positive definite if its Hermitian part is positive definite, i.e., if

$$x^H (M_c(a) + M^H_c(a)) x = x^H H^H (C(a)|_{t_i=1} + C^H(a)|_{t_i=1}) H x = \tilde{x}^H (C(a)|_{t_i=1} + C^H(a)|_{t_i=1}) \tilde{x} > 0$$  \hspace{1cm} (B.4)

for any $x \neq 0 \in \mathbb{C}^{P \times 1}$ and $\tilde{x} = H x$. In the above equation, we assume that $H$ has full column-rank. Hence, we can conclude that the Hermitian part of $M_c(a)$ is positive definite if the Hermitian matrix $\tilde{C}(a) = C(a)|_{t_i=1} + C^H(a)|_{t_i=1}$ is positive definite. Therefore, it suffices to derive the conditions under which $\tilde{C}(a)$ is positive definite. We write $\tilde{C}(a)$ as (B.5), shown at the bottom of the page, which can be viewed as a matrix composed of $(K - 1)$ similar nested matrices, denoted by $\tilde{C}^{(k)}(a)$, for $k = 1, \ldots, K - 1$. The matrix $\tilde{C}^{(k)}(a)$ is obtained from $\tilde{C}^{(K-1)}(a)$ by removing the last row and the last column and appending one column and one row of zeros from the left and the top, so that the dimension $(K - 1) \times (K - 1)$ of the matrix remains unchanged. Thus $\tilde{C}(a)$ can be expressed as

$$\tilde{C}(a) = \tilde{C}^{(1)}(a) + \tilde{C}^{(2)}(a) + \cdots + \tilde{C}^{(K-1)}(a)$$  \hspace{1cm} (B.6)

where

$$\tilde{C}^{(k)}(a) \triangleq J_{k-1} \tilde{C}^{(1)}(a) J_{k-1}^T, \hspace{1cm} J_k = \begin{bmatrix} 0_{k,K-1-k} & 0_{k,k} \\ I_{K-1-k} & 0_{K-1-k,k} \end{bmatrix}$$  \hspace{1cm} (B.7)

and $\tilde{C}^{(1)}(a)$ is defined in (B.9), at the bottom of the page. In order to obtain the conditions for which $\tilde{C}(a)$ is positive definite, due to the nested structure of $\tilde{C}(a)$ in (B.6) and (B.7), it is sufficient to show that $\tilde{C}^{(1)}(a)$ is positive definite, as in this case the matrices $\tilde{C}^{(k)}(a)$ are non-negative definite for $k = 1, \ldots, K - 1$. Applying the elementary matrix operations of adding the $(k+1)$th column multiplied by $-a$ to the $k$th column, and then adding the $(k+1)$th row multiplied by $-a^*$ to the $k$th row, for $k = 1, \ldots, K - 1$ we obtain

$$\tilde{C}^{(1)}(a) = W^H(a)\tilde{C}^{(1)}(a)W(a)$$  \hspace{1cm} (B.10)

$$\begin{bmatrix} c_1 \\ c_2 a \\ c_3 a^2 \\ \vdots \\ c_{K-1} a^{K-2} \\ \left(\sum_{\ell=1}^{2} c_{K-\ell}\right) a^{K-3} \\ \left(\sum_{\ell=1}^{2} c_{K-\ell}\right) a^{K-4} \\ \left(\sum_{\ell=1}^{K-1} c_{K-\ell}\right) a^{K-5} \end{bmatrix}$$  \hspace{1cm} (B.2)

$$\tilde{C}(a) = \begin{bmatrix} 2c_1 & c_2 a^* & c_3 (a^*)^2 & \cdots & c_{K-1} (a^*)^{K-2} \\ c_2 a & 2(c_1 + c_2) & (c_2 + c_3)a^* & \cdots & (c_{K-2} + c_{K-1})(a^*)^{K-3} \\ c_3 a^2 & (c_2 + c_3)a & 2(c_1 + c_2 + c_3) & \cdots & (\sum_{k=K-3}^{K-1} c_k)(a^*)^{K-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{K-1} a^{K-2} & (c_{K-2} + c_{K-1})a^{K-3} & \left(\sum_{k=K-3}^{K-1} c_k\right) a^{K-4} & \cdots & 2(\sum_{k=K-1}^{K-1} c_k) \end{bmatrix}$$  \hspace{1cm} (B.5)

$$\tilde{C}^{(1)}(a) \triangleq \begin{bmatrix} 2c_1 & c_2 a^* & c_3 (a^*)^2 & \cdots & c_{K-1} (a^*)^{K-2} \\ c_2 a & 2c_2 & c_3 a^* & \cdots & c_{K-1} (a^*)^{K-3} \\ c_3 a^2 & c_3 a & 2c_3 & \cdots & c_{K-1} (a^*)^{K-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{K-1} a^{K-2} & c_{K-1} a^{K-3} & c_{K-1} a^{K-4} & \cdots & 2c_{K-1} \end{bmatrix}$$  \hspace{1cm} (B.9)
where $W(a)$ is a $(K - 1) \times (K - 1)$ full-rank square matrix defined as

$$
W(a) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
-a & 1 & 0 & \cdots & 0 \\
0 & -a & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -a & 1
\end{bmatrix}.  \tag{B.11}
$$

It can readily be verified that according to (B.10), we obtain $\hat{C}^{(1)}(a)$ in (B.12), at the top of the page. Next we derive conditions on the coefficients $c_k$, for $k = 1, \ldots, K - 1$, for which $\hat{C}^{(1)}(a)$ is positive definite, as this then implies the positive definiteness property of $\bar{C}(a)$. Applying the Gershgorin circle theorem to the rows of the above matrix yields

$$
|\lambda_1 - 2c_2| \leq -c_2 a^*  \tag{B.13}
$$

$$
|\lambda_{K-1} - 2c_{K-1}| \leq -c_{K-1} a^*  \tag{B.14}
$$

$$
|\lambda_i - 2c_i| \leq c_i a + -c_{i+1} a^*  \tag{B.15}
$$

for $i = 2, \ldots, K - 2$, where $\lambda_i$ is the $i$th eigenvalue of the Hermitian matrix $\hat{C}^{(1)}(a)$ and, hence, $\lambda_i$ is real-valued. Expanding the above inequalities and taking into account that $c_1$ is a real positive scalar, we obtain

$$
2c_1 - c_2 |a| \leq \lambda_1 \leq 2c_1 + c_2 |a|  \tag{B.16}
$$

$$
2c_{K-1} - c_{K-1} |a| \leq \lambda_{K-1} \leq 2c_{K-1} + c_{K-1} |a|  \tag{B.17}
$$

$$
c_i(2 - |a|) - c_{i+1} \leq \lambda_i \leq c_i(2 + |a|) + c_{i+1}.  \tag{B.18}
$$

To guarantee that $\hat{C}^{(1)}(a)$ is positive definite, $\lambda_i$ must be positive. Hence, the values $2c_1 - c_2 |a|$ and $c_i(2 - |a|) - c_{i+1}$ need to be positive. Assuming $|a| \leq 1$, we can remark that if $c_1 \geq c_{i+1}$ and $2c_1 \geq c_2$, then $c_i(2 - |a|) - c_{i+1}$ is positive and, hence, $\lambda_i$ will always be positive. Therefore, for $a \leq 1$, and provided that $2c_1 > c_2 > \cdots > c_{K-1} > 0$, the matrix $\hat{C}^{(1)}(a)$ is positive definite. This completes our proof.

**APPENDIX C**

**CRAMÉR-RAO BOUND DERIVATION**

Our derivation of the Cramér-Rao bound (CRB) for the harmonic retrieval problem in the incomplete sample case is based on [38]. Using the stochastic signal model

$$
y(n) = \sum_{p=1}^{P} a_p x_p(n) + z(n)  \tag{C.1}
$$

and assuming that the noise has Gaussian distribution, $y(n)$ has a zero-mean Gaussian distribution with the covariance matrix $\mathbf{R}$. We define the vector $\omega$

$$
\omega = [\text{Re}\{a_1\}, \ldots, \text{Re}\{a_P\}, \text{Im}\{a_1\}, \ldots, \text{Im}\{a_P\}]  \tag{C.2}
$$

In our model, we assumed that the noise power $\sigma^2$, the signal covariance matrix $\mathbf{R}_s$ and the vector $\omega$ are unknown.

Defining $T$ as in (7), it can be shown from [38] that the compact form of the CRB matrix for the unknown vector $\omega$ can be expressed as

$$
\text{CRB}(\omega) = \frac{\sigma^2}{2N} \left[ \text{Re}\left\{ (1 \otimes U) \circ (D^H \mathbf{P}^\perp \mathbf{D}) \right\} \right]^{-1}  \tag{C.3}
$$

where $1$ denotes the $2 \times 2$ matrix containing ones in all entries, the modified signal generator matrix is defined as $\mathbf{H} \triangleq \mathbf{T}^H \mathbf{H}$, and $\mathbf{P}^\perp = \mathbf{I} - \mathbf{H} (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H$.

The $2P \times K$ matrix $\mathbf{D}$ indicates the first partial derivative of the columns of $\mathbf{H}$ with respect to the corresponding unknown parameters of vector $\omega$

$$
\mathbf{D} = \begin{bmatrix}
\frac{\partial \tilde{a}_1}{\partial \text{Re}\{a_1\}}, & \cdots, & \frac{\partial \tilde{a}_P}{\partial \text{Re}\{a_P\}}, & \frac{\partial \tilde{a}_1}{\partial \text{Im}\{a_1\}}, & \cdots, & \frac{\partial \tilde{a}_P}{\partial \text{Im}\{a_P\}}
\end{bmatrix}  \tag{C.6}
$$

where $\tilde{a}_p$ is the $p$th column of $\mathbf{H}$ given as

$$
\tilde{a}_p \triangleq T [1, a_p, a_p^2, \ldots, a_p^{K-1}]^T.  \tag{C.7}
$$

From (C.6), the matrix $\mathbf{D}$ can be written as

$$
\mathbf{D} = \mathbf{Q} \mathbf{H}_i \mathbf{j} \mathbf{Q} \mathbf{H}_i^H  \tag{C.8}
$$

where

$$
\mathbf{Q} \triangleq \text{diag}(0, 1, 2, \ldots, K - 1).  \tag{C.9}
$$

Then, we can compute the required $2P \times 2P$ matrix CRB$(\omega)$ from (C.3).

**REFERENCES**


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